Internal Acoustic Gravity Waves *

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(Z. Naturforsch. 23a, 1459—1470 [1968]; received 2 March 1968)

An exhaustive analysis of wave motion in a compressible isothermal medium under the influence of gravity is presented. The dispersion relation governing the wave propagation is derived from the linearized equations of fluid dynamics and thermodynamics, and it is arranged in a non-dimensional form. Penetration depths, frequency cutoffs, and particle orbits are calculated under the assumption of an ideal gas. With the nondimensional form of the dispersion relation these data can be expressed in a form independent of the constants describing a particular atmosphere. The results can be conventiently displayed on a number of diagrams valid for monatomic and diatomic gases. Dissipative effects arising from viscosity and heat conduction are neglected.

1. Introduction

Internal gravity waves have been discussed previously in some detail in regard to investigating certain phenomena such as the irregular motion in the earth's higher atmosphere [1], and the heating of the solar corona [2]. These waves are also of considerable significance in geophysics for the internal wave motions within the oceans and the earth's core [3]. LANDAU has considered the problem of internal gravity waves in an incompressible fluid [4]. We also would like to refer to some earlier work in this field [5]. It can be shown that in compressible media under the influence of gravity two modes of wave propagation exist. The purpose of this paper is to give a detailed analysis of these wave modes. Possible mechanisms for the generation of internal gravity waves will not be discussed. We restrict ourselves to waves of infinitesimal amplitude and neglect dissipative effects arising from viscosity and heat conduction.

2. Fundamental Equations

The fundamental equations necessary to describe the wave motions are:

1. *Euler equation*

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} = -\frac{\triangle p}{\varrho} + \boldsymbol{g}\,. \tag{2.1}$$

In this equation $\boldsymbol{v}$ is the velocity vector of a particle of the oscillating medium, p is the pressure, ϱ the density, and $\boldsymbol{g}$ the gravity vector. For convenience we choose $\boldsymbol{g}$ to be along the negative z direction (downward):

$$\boldsymbol{g} = -g\, \boldsymbol{e}_z\,, \tag{2.2}$$

where $\boldsymbol{e}_z$ is the unit vector in the z direction.

2. *Continuity equation:*

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho\, \boldsymbol{v}) = 0\,. \tag{2.3}$$

3. *Second law of thermodynamics:* (adiabatic approximation)

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \nabla s = 0\,, \tag{2.4}$$

where s is the specific entropy.

4. *Equation of state:*

$$\varrho = \varrho(p, s)\,. \tag{2.5}$$

Thus, we have for the total differential

$$\mathrm{d}\varrho = \left(\frac{\partial \varrho}{\partial s}\right)_p \mathrm{d}s + \left(\frac{\partial \varrho}{\partial p}\right)_s \mathrm{d}p\,. \tag{2.6}$$

We will assume that for the equilibrium conditions the variation of density and pressure, with height is exponential, as is the case for an ideal gas. Thus, we have

$$\varrho_{\mathrm{eq}} = \varrho_0\, e^{-z/h} \quad \text{and} \quad p_{\mathrm{eq}} = p_0\, e^{-z/h}\,, \tag{2.7}$$

* This work has been suported in part by the National Aeronautics and Space Administration under Grant No. NGR-29-001-016.

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where ϱ_0 and p_0 are constants, and h is the so called "scale height".

3. The Linearized Equations of Motion

We choose our coordinate system in such a way that the wave propagation vector is in the x-z plane and assume all quantities are independent of the y coordinate. Hence, we may put to first order

$$\begin{aligned} \varrho &= [\varrho_0 + \varrho'(x,z,t)]\, e^{-z/h}\,, \\ p &= [p_0 + p'(x,z,t)]\, e^{-z/h}\,, \\ s &= s + s'(x,z,t)\,. \end{aligned} \tag{3.1}$$

The quantities ϱ', p', and s', together with the velocity $\boldsymbol{v} = \boldsymbol{v}\,(x, z, t)$, are considered to be small quantities of first order.

For equilibrium conditions, consider the Euler equation (2.1) to be of zeroth order:

$$0 = -\frac{\triangle p_{\mathrm{eq}}}{\varrho_{\mathrm{eq}}} + \boldsymbol{g}\,. \tag{3.2}$$

By combining Eq. (2.7) with (3.2), we have

$$\boldsymbol{g} = \frac{\triangle [p_0\, e^{-z/h}]}{\varrho_0 e^{-z/h}} = -\frac{p_0\, \boldsymbol{e}_z}{h\,\varrho_0}\,. \tag{3.3}$$

With the aid of (3.1) and (3.3), there results the linearized Euler equation

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{\triangle p'}{\varrho_0} + \frac{p'}{h\,\varrho_0}\boldsymbol{e}_z + \frac{\varrho'}{\varrho_0}\boldsymbol{g}\,. \tag{3.4}$$

With the help of (3.1), and keeping only first order terms, the continuity equation becomes

$$\frac{\partial \varrho'}{\partial t} + \varrho_0 \nabla \cdot \boldsymbol{v} - \frac{\varrho_0}{h}(\boldsymbol{v}\cdot\boldsymbol{e}_z) = 0\,. \tag{3.5}$$

It is easy to see that the second law of thermodynamics can be written as follows,

$$\frac{\partial s'}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{e}_z)\frac{\mathrm{d}s_{\mathrm{eq}}}{\mathrm{d}z} = 0\,. \tag{3.6}$$

To evaluate the equation of state, we observed that the density is a continuous function of both the entropy and pressure, so that we may perform a Taylor expansion about the equilibrium state. Keeping only first order terms, we have (cancelling the "height" exponentials),

$$\varrho = \varrho_0 + \left(\frac{\partial \varrho_0}{\partial s_0}\right)_p s' + \left(\frac{\partial \varrho_0}{\partial p_0}\right) p'\,.$$

By substitution, we have

$$\varrho' = \left(\frac{\partial \varrho_0}{\partial s_0}\right)_p s' + \frac{1}{a^2}\,p'\,. \tag{3.7}$$

In Eq. (3.7), $a = [(\partial p_0/\partial \varrho_0)_s]^{1/2}$ is known as the sound velocity in a compressible medium. Eqs. (3.4) —(3.7) form a coupled set of linear differential equations for the unknowns $\boldsymbol{v}$, ϱ', p', and s'.

In order to obtain one equation for the velocity, we first take the time derivative of Eqs. (3.4) and (3.7). This gives, respectively,

$$\frac{\partial^2 \boldsymbol{v}}{\partial t^2} + \frac{1}{\varrho_0}\nabla\frac{\partial p'}{\partial t} - \frac{1}{h\,\varrho_0}\frac{\partial p'}{\partial t}\boldsymbol{e}_z - \frac{1}{\varrho_0}\frac{\partial \varrho'}{\partial t}\boldsymbol{g} = 0 \tag{3.8}$$

and

$$\frac{\partial \varrho'}{\partial t} - \left(\frac{\partial \varrho_0}{\partial s_0}\right)_p \frac{\partial s'}{\partial t} - \frac{1}{a^2}\frac{\partial p'}{\partial t} = 0\,. \tag{3.9}$$

Then we reduce the coupled set of Eqs. (3.5), (3.6), (3.8), and (3.9) by eliminating the unknowns, resulting in only one equation for $\boldsymbol{v}$. For example, solving Eq. (3.5) for $\partial\varrho'/\partial t$, Eq. (3.6) for $\partial s'/\partial t$, and Eq. (3.9) for $\partial p'/\partial t$; substituting $\partial\varrho'/\partial t$ of (3.5) and $\partial s'/\partial t$ of (3.6) into Eq. (3.9), and then substituting $\partial p'/\partial t$ of (3.9) and $\partial p'/\partial t$ of (3.5) into Eq. (3.8), we have for the equation of $\boldsymbol{v}$

$$\begin{aligned} &\frac{\partial^2 \boldsymbol{v}}{\partial t^2} + \frac{a^2}{h}\nabla(\boldsymbol{v}\cdot\boldsymbol{e}_z) - a^2\nabla(\nabla\cdot\boldsymbol{v}) + \nabla\left[\frac{a^2(\boldsymbol{v}\cdot\boldsymbol{e}_z)}{\varrho_0}\left(\frac{\partial\varrho_0}{\partial s_0}\right)_p \frac{\mathrm{d}s_{\mathrm{eq}}}{dz}\right] \\ &- \frac{a^2}{h^2}(\boldsymbol{v}\cdot\boldsymbol{e}_z)\,\boldsymbol{e}_z + \frac{a^2}{h}(\nabla\cdot\boldsymbol{v})\,\boldsymbol{e}_z - \frac{a^2(\boldsymbol{v}\cdot\boldsymbol{e}_z)}{h\,\varrho_0}\left(\frac{\partial\varrho_0}{\partial s_0}\right)_p \frac{\mathrm{d}s_{\mathrm{eq}}}{\mathrm{d}z}\boldsymbol{e}_z - \frac{(\boldsymbol{v}\cdot\boldsymbol{e}_z)}{h}\boldsymbol{g} + (\nabla\cdot\boldsymbol{v})\,\boldsymbol{g} = 0\,. \end{aligned} \tag{3.10}$$

Vector Eq. (3.10) represents a set of two linear homogeneous differential equations for the unknowns v_x and v_z.

4. Perfect Gas Assumption

It is quite obvious that the case of a perfect gas deserves special consideration for the reason of being a very good approximation in most situations and for its mathematical simplicity.

For a perfect gas under adiabatic compression, we have:

$$p\,\varrho^{-\gamma} = p_0\,\varrho_0^{-\gamma}\,, \tag{4.1}$$

where $\gamma = c_p/c_v$ is the ratio of the specific heats, at constant pressure and density. For a monatomic gas, one has $\gamma = 5/3 = 1.6667$, and for a diatomic gas, $\gamma = 7/5 = 1.4$.

From Eq. (4.1) we have for the sound velocity

$$a^2 = \mathrm{d}p/\mathrm{d}\varrho = \gamma\,(p_0/\varrho_0)\,. \tag{4.2}$$

With the help of (3.3) and (4.2), we express the scale height as follows

$$h = a^2/g\gamma\,. \tag{4.3}$$

To express the term in the square brackets of Eq. (3.10) for the case of an ideal gas, we consider the well known thermodynamic relations,

$$\left(\frac{\partial s}{\partial p}\right)_T = \frac{1}{\varrho^2}\left(\frac{\partial \varrho}{\partial T}\right)_p, \quad \text{and} \left(\frac{\partial \varrho}{\partial s}\right)_p = \frac{T}{c_p}\left(\frac{\partial \varrho}{\partial T}\right)_p, \tag{4.4}$$

where T is the absolute temperature. By making use of Eq. (4.4) we can write

$$\begin{aligned}\frac{a^2}{\varrho_0}\left(\frac{\partial \varrho_0}{\partial s_0}\right)_p \frac{\mathrm{d}s_{\mathrm{eq}}}{\mathrm{d}z} &= \frac{a^2}{\varrho_0}\left[\frac{T}{c_p}\left(\frac{\partial \varrho}{\partial T}\right)_p\right]\left[-\frac{g}{\varrho}\left(\frac{\partial \varrho}{\partial T}\right)_p\right] \\ &= -\frac{T}{c_p}\,a^2\,\frac{g}{\varrho^2}\left[\left(\frac{\partial \varrho}{\partial T}\right)_p\right]^2 = -\frac{g}{c_p T}\,a^2\,.\end{aligned} \tag{4.5}$$

It is expedient to put the velocity of sound in still another form. From the perfect gas assumption we have

$$c_p - c_v = R/A = c_v(\gamma - 1)\,, \tag{4.6}$$

where R is the gas constant, and A is the molecular weight. Thus, from Eq. (4.2), we obtain

$$a^2 = \gamma(R/A)\,T = (\gamma - 1)\,c_p\,T\,. \tag{4.7}$$

With Eq. (4.5) and (4.7), the term in the square brackets in Eq. (3.10) becomes

$$\frac{a^2}{\varrho_0}(\boldsymbol{v}\cdot\boldsymbol{e}_z)\left(\frac{\partial \varrho_0}{\partial s_0}\right)_p \frac{\mathrm{d}s_{\mathrm{eq}}}{\mathrm{d}z} = -\,g(\gamma - 1)\,(\boldsymbol{v}\cdot\boldsymbol{e}_z)\,. \tag{4.8}$$

By substituting Eq. (4.8) into Eq. (3.10), we have the equation for $\boldsymbol{v}$ for internal gravity waves within a perfect gas:

$$\begin{aligned}\frac{\partial^2 \boldsymbol{v}}{\partial t^2} - a^2\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{v} - \mathrm{grad}\,(\boldsymbol{v}\cdot\boldsymbol{g}) \\ -\,(\gamma - 1)\,(\mathrm{div}\,\boldsymbol{v})\,\boldsymbol{g} = 0\,.\end{aligned} \tag{4.9}$$

5. Plane Wave Analysis

We expand the solution of Eq. (4.9) into plane waves by putting:

$$\boldsymbol{v}(x, z, t) = \boldsymbol{v}\exp\{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\}\,, \tag{5.1}$$

where $\boldsymbol{v}$ is a constant vector.

By the substitution of (5.1) into (4.9) and performing the indicated mathematical operations, we obtain the relation

$$\begin{aligned}\omega^2\boldsymbol{v} - [a^2(\boldsymbol{k}\cdot\boldsymbol{v}) - i(\boldsymbol{v}\cdot\boldsymbol{g})]\,\boldsymbol{k} \\ +\,i(\gamma - 1)\,(\boldsymbol{k}\cdot\boldsymbol{v})\,\boldsymbol{g} = 0\,.\end{aligned} \tag{5.2}$$

This vector equation represents a set of two linear homogeneous equations for the unknowns v_x and v_z. The condition for a solution, the vanishing of the determinant of the coefficients, is the dispersion relation.

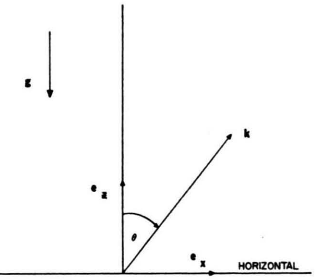

Fig. 1. The coordinate system and the direction of the vector $\boldsymbol{g}$ and $\boldsymbol{k}$ with respect to the horizontal.

6. The Dispersion Relation

The $\boldsymbol{e}_x$ and $\boldsymbol{e}_z$ components of Eq. (5.2) are:

$$\begin{aligned}v_x(\omega^2 - a^2 k_x^2) \\ +\,v_z[-\,a^2 k_x k_z - i g\gamma k_x + i g(\gamma - 1)\,k_x] = 0\,,\end{aligned} \tag{6.1}$$

$$\begin{aligned}v_x[-\,a^2 k_x k_z - i g(\gamma - 1)\,k_x] \\ +\,v_z[\omega^2 - a^2 k_z^2 - i g(\gamma - 1)\,k_z - i g k_z] = 0\,.\end{aligned} \tag{6.2}$$

Setting the determinant of the coefficients of (6.1)—(6.2) to zero, we have the dispersion relation

$$\begin{vmatrix} \omega^2 - a^2 k_x^2 & -a^2 k_x k_z - i g k_x \\ -a^2 k_x k_z - i g(\gamma - 1)\,k_x & \omega^2 - a^2 k_z^2 - i g\gamma k_z \end{vmatrix} = 0\,, \tag{6.3}$$

which can be brought into the following form:

$$\begin{aligned}\omega^4 - \omega^2 a^2(k_x^2 + k_z^2) \\ +\,g^2(\gamma - 1)\,k_x^2 - i\,\omega^2 g\gamma k_z = 0\,.\end{aligned} \tag{6.4}$$

We know that the quantities p and ϱ, decrease exponentially in the z direction, causing the amplitude of the waves to increase exponentially in this direction. This behavior permits us to remove the imaginary term in the wave vector, $\boldsymbol{k}$, by merely changing variables. We put

$$\begin{aligned}k_x &= K_x\,, \\ k_z &= K_z - \frac{i}{2h} = K_z - \frac{i g\gamma}{2a^2}\,,\end{aligned} \tag{6.5}$$

and we substitute this into Eq. (6.4). We have

$$\begin{aligned}\omega^4 - \omega^2 a^2(K_x^2 + K_z^2) \\ +\,g^2(\gamma - 1)\,K_x^2 - \frac{\omega^2 g^2\gamma^2}{4a^2} = 0\,,\end{aligned} \tag{6.6}$$

which is the form of the dispersion relation used by HINES[1]. In the absence of gravity, this expression reduces to

$$\omega^2 = a^2 K^2 , \tag{6.7}$$

the dispersion relation valid for sound waves.

It is convenient to introduce the angle θ, between the z-direction and the direction of the wave propagation vector. We thus have

$$K_x = K \sin\theta ,$$
$$K_z = K \cos\theta ,$$
and
$$K^2 = K_x^2 + K_z^2 . \tag{6.8}$$

By the substitution of (6.8) into (6.6), we have

$$\omega^4 - \omega^2 a^2 \left(K^2 + \frac{g\gamma^2}{4a^4}\right) + g^2(\gamma-1) K^2 \sin^2\theta = 0 . \tag{6.9}$$

Solving (6.9) for ω^2, there results:

$$\omega_\pm^2 = \tfrac{1}{2}\left[a^2\left(K^2 + \frac{g^2\gamma^2}{4a^4}\right)\right] \pm \tfrac{1}{2}\left[a^4\left(K^2 + \frac{g^2\gamma^2}{4a^4}\right)^2 - 4g^2(\gamma-1) K^2 \sin^2\theta\right]^{\frac{1}{2}}, \tag{6.10}$$

or solving for K

$$K_\pm = \pm\, \omega \left[\frac{\omega^2 - g^2\gamma^2/(4a^2)}{\omega^2 a^2 - g^2(\gamma-1)\sin^2\theta}\right]^{\frac{1}{2}}. \tag{6.11}$$

In order to put these equations into dimensionless form, we must first define a characteristic wave number K_c and a characteristic frequency ω_c. It is convenient to choose

$$K_c \equiv g/a^2 \quad \text{and} \quad \omega_c \equiv g/a . \tag{6.12}$$

Eq. (6.10) can then be written in the following form

$$W_\pm^2 = (\tfrac{1}{2}\varkappa^2 + \tfrac{1}{8}\gamma^2) \pm \tfrac{1}{2}[\varkappa^4 + \varkappa^2\{\tfrac{1}{2}\gamma^2 - 4(\gamma-1)\sin^2\theta\} + \tfrac{1}{16}\gamma^4]^{\frac{1}{2}}, \tag{6.13}$$

or

$$\varkappa_\pm = W\left[\frac{W^2 - \frac{1}{4}\gamma^2}{W^2 - (\gamma-1)\sin^2\theta}\right]^{\frac{1}{2}}, \tag{6.14}$$

where

$$W \equiv \omega/\omega_c \quad \text{and} \quad \varkappa \equiv K/K_c . \tag{6.15}$$

The dispersion relation according to Eq. (6.13) is drawn in Fig. 2 and 3 for monatomic gases and in Fig. 4 and 5 for diatomic gases. From the diagrams it can be seen that there exists a gap of forbidden frequencies, which will be discussed in detail below.

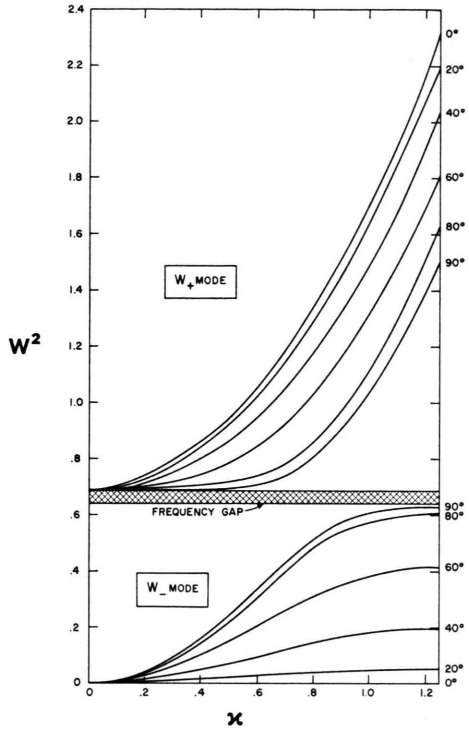

Fig. 2. The dispersion relation $W^2 = W^2(\varkappa)$ for monatomic gases.

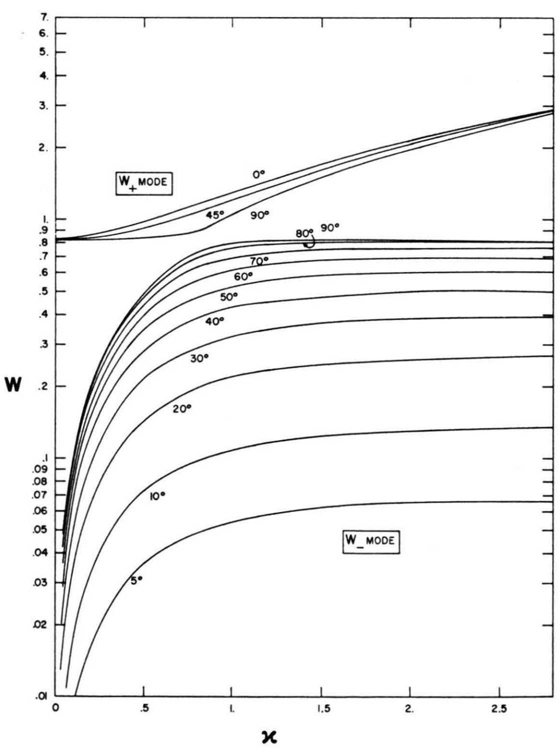

Fig. 3. The dispersion relation $W = W(\varkappa)$ for monatomic gases.

7. Penetration Depths

For a particular frequency W, a wave may, in general, be totally, partially, or not at all attenuated, depending on the angle of propagation, θ. The wave number is pure imaginary if the term under the

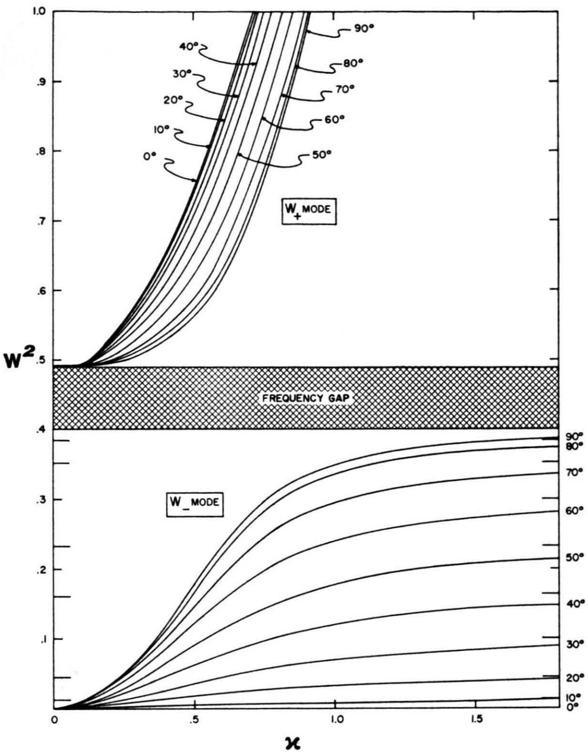

Fig. 4. The dispersion relation $W^2 = W^2(\varkappa)$ for diatomic gases.

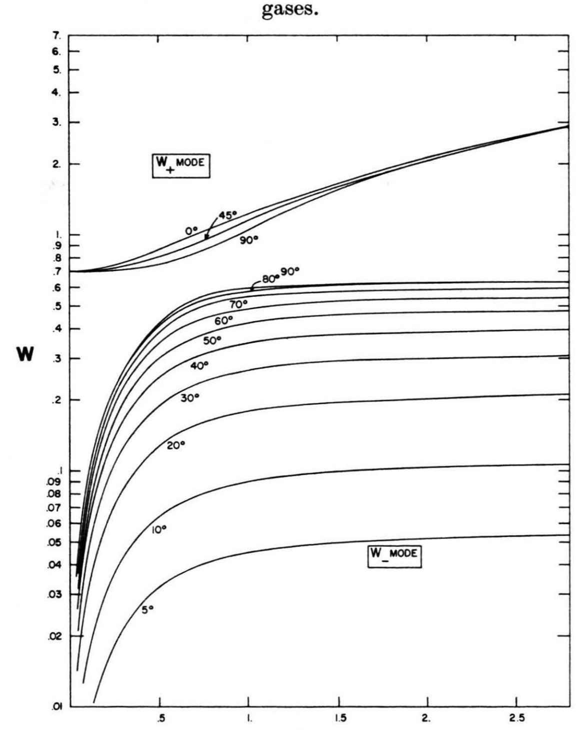

Fig. 5. The dispersion relation $W = W(\varkappa)$ for diatomic gases.

square root in the dispersion relation (6.14) is negative. This may occur in either of two ways.

First, let us assume $W > \gamma/2$, then the condition for $\mathrm{Re}(\varkappa) = 0$, that is, $\varkappa$ having an imaginary part only, is given by

$$[W^2 - (\gamma - 1)\sin^2\theta] < 0\,. \qquad (7.1)$$

Under the given condition, W has a minimum value of $\gamma/2$; thus, the inequality (7.1) becomes

$$\sin^2\theta > \tfrac{1}{4}\gamma^2/(\gamma - 1) > 1\,. \qquad (7.2)$$

Clearly the inequality cannot be satisfied for real values of θ. Thus, for values of the frequency greater than $\gamma/2$, we find that the wave propagation is unattenuated.

Second, let us assume $W < \gamma/2$, then the condition for $\mathrm{Re}(\varkappa) = 0$ is given by

$$[W^2 - (\gamma - 1)\sin^2\theta] > 0\,. \qquad (7.3)$$

From this, we see that we have total attenuation for frequencies $W < \gamma/2$, if

$$|\sin\theta| < W/(\gamma - 1)^{\frac{1}{2}}\,. \qquad (7.4)$$

Then, the penetration depth, δ, may be obtained from Eq. (6.14), under the restriction of the inequality (7.4). We thus have

$$\delta = \frac{1}{\mathrm{Im}(\varkappa)} = \frac{1}{W}\left[\frac{W^2 - (\gamma - 1)\sin^2\theta}{\frac{1}{4}\gamma^2 - W^2}\right]^{\frac{1}{2}}. \qquad (7.5)$$

The penetration depth according to Eq. (7.5) is plotted in Fig. 6 and 7 for monatomic gases and Fig. 8 and 9 for diatomic gases.

On the other hand, it can be shown that for any given real wave number $\varkappa$ the frequency W is always real, implying no attenuation in Time.

The existence of an imaginary part of $W_\pm$ is obtained from the dispersion relation (6.13) and is given by

$$\{\varkappa^4 + \varkappa^2[\tfrac{1}{2}\gamma^2 - 4(\gamma - 1)\sin^2\theta] + \tfrac{1}{16}\gamma^4\} < 0\,, \qquad (7.6)$$

which may be reduced to:

$$|\sin\theta| > \frac{\varkappa^2 + \frac{1}{4}\gamma^2}{2\varkappa(\gamma - 1)^{1/2}}\,. \qquad (7.7)$$

Choosing the maximum of $|\sin\theta| = 1$, we have the condition,

$$1 > \frac{\varkappa^2 + \frac{1}{4}\gamma^2}{2\varkappa(\gamma - 1)^{1/2}} \equiv f(\varkappa)\,. \qquad (7.8)$$

The function $f(\varkappa)$ is of the form

$$f(\varkappa) = A\varkappa + B/\varkappa\,, \qquad (7.9)$$

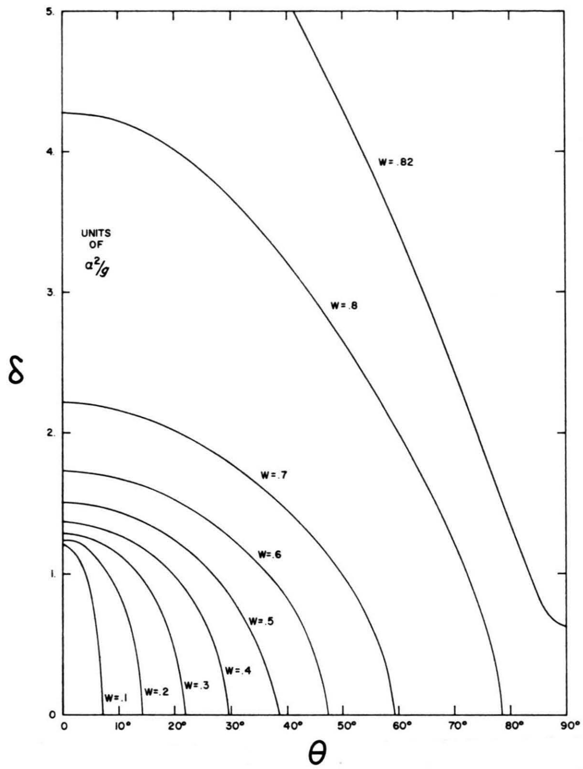

Fig. 6. Penetration depths for monatomic gases.

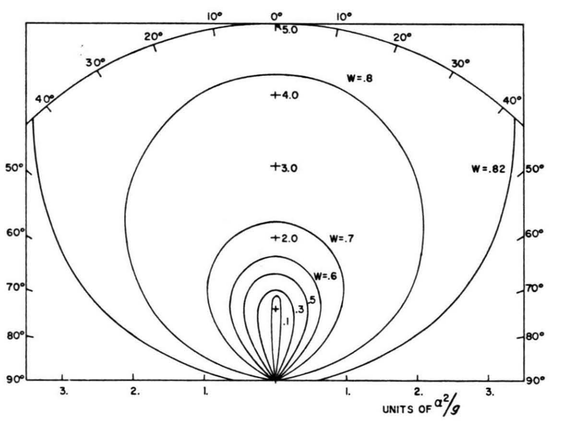

Fig. 7. Polar diagram for penetration depth for monatomic gases.

where the constants A and B depend only upon γ. $f(\varkappa)$ has a minimum at the value

$$\varkappa_{\min} = \tfrac{1}{2}\gamma\,. \tag{7.10}$$

Choosing γ either for a monatomic gas or for a diatomic gas, and substituting Eq. (7.10) into the inequality (7.8), we see that the inequality cannot be satisfied, implying that condition (7.7) will not hold for any real value of θ.

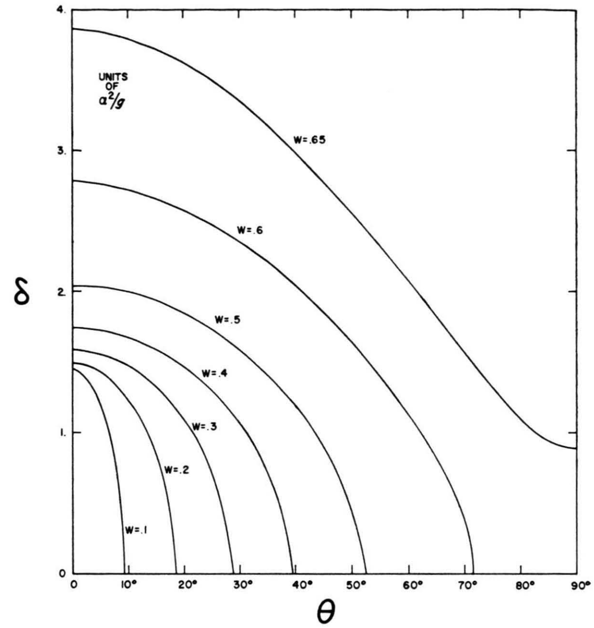

Fig. 8. Penetration depth for diatomic gases.

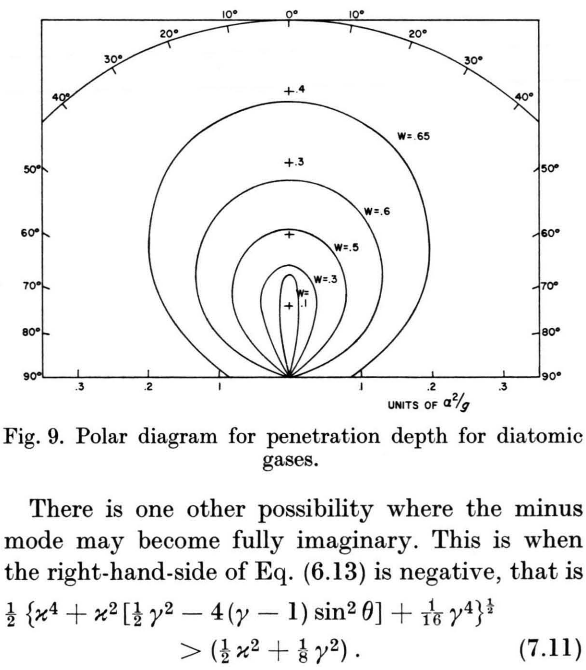

Fig. 9. Polar diagram for penetration depth for diatomic gases.

There is one other possibility where the minus mode may become fully imaginary. This is when the right-hand-side of Eq. (6.13) is negative, that is

$$\tfrac{1}{2}\left\{\varkappa^4 + \varkappa^2\left[\tfrac{1}{2}\gamma^2 - 4(\gamma-1)\sin^2\theta\right] + \tfrac{1}{16}\gamma^4\right\}^{\frac{1}{2}} > \left(\tfrac{1}{2}\varkappa^2 + \tfrac{1}{8}\gamma^2\right). \tag{7.11}$$

Upon squaring both sides and rearranging, we get

$$\sin^2\theta < 0\,,$$

which is again not possible for real values of θ.

Hence, we may conclude that the W_+ and W_- modes are entirely real for all real values of $\varkappa$ and θ, and for both diatomic and monatomic gases.

8. Frequency Cutoffs

From the dispersion relation (6.13) where the frequency, $W_{\pm}$, is a function of the wave number $\varkappa$, we note that the domain of $W_{\pm}$ is limited within the range $0 < \varkappa < \infty$. This implies that there may be an upper and lower limit or "cutoff" for each wave mode.

First, let us consider the W_{+} mode. This mode has no upper cutoff, because as $\varkappa$ goes to infinity, W_{+} also goes to infinity. However, as $\varkappa$ goes to zero, W_{+} approaches the limiting value of $\gamma/2$, thus the W_{+} mode has a lower frequency cutoff equal to $\gamma/2$.

Second, let us consider the W_{-} mode. This mode has no lower cutoff, because as $\varkappa$ goes to zero, W_{-} also goes to zero. To find W_{-} as $\varkappa$ goes to infinity, we make a binomial expansion of the right-hand-side of Eq. (6.13). Upon expansion, we have

$$W_{-}^{2} \approx \varkappa^{2}\left[\frac{1}{2} + \frac{\gamma^{2}}{8\varkappa^{2}} - \frac{1}{4}\left(\frac{A}{\varkappa^{2}} + \frac{\gamma^{4}}{16\varkappa^{4}}\right)\right], \qquad (8.1)$$

where $A \equiv \frac{1}{2}\gamma^{2} - 4(\gamma - 1)\sin^{2}\theta$.
Thus, we have

$$W_{-}^{2} \approx \varkappa^{2}\left(\frac{\gamma^{2}}{8\varkappa^{2}} - \frac{A}{4\varkappa^{2}} - \frac{\gamma^{4}}{16\varkappa^{4}}\right) = (\gamma - 1)\sin^{2}\theta - \frac{\gamma^{4}}{64\varkappa^{4}} \qquad (8.2)$$

In the limit $\varkappa \to \infty$, the cutoff angle is determined by

$$W_{-} = (\gamma - 1)^{\frac{1}{2}}\sin\theta_{\mathrm{c}}. \qquad (8.3)$$

This is the upper cutoff for the W_{-} mode.

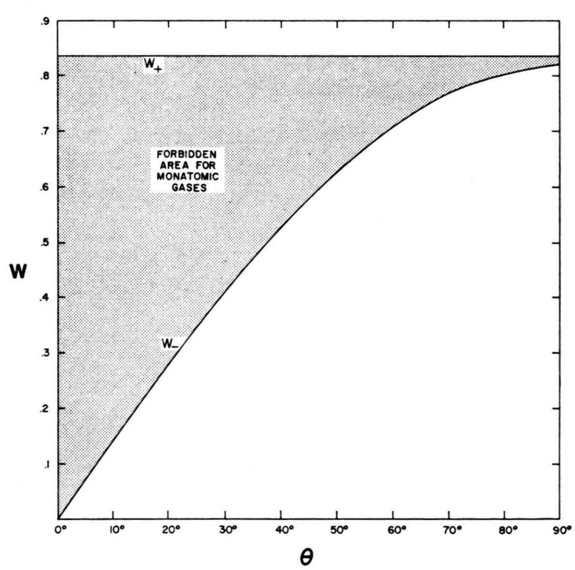

Fig. 10. Frequency cutoffs for monatomic gases.

These limits are plotted versus propagation angle θ in Fig. 10 for monatomic gases and in Fig. 11 for diatomic gases.

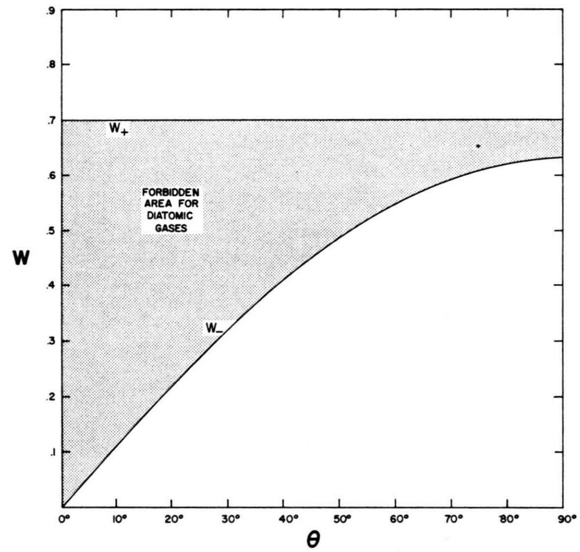

Fig. 11. Frequency cutoffs for diatomic gases.

Another way of understanding the result is to say that for a given frequency there exists for the W_{-} mode a vertical cone determined by the cutoff angle θ_{c} of Eq. (8.3), within which wave propagation is attenuated. ("Forbidden cone", see Fig. 12.)

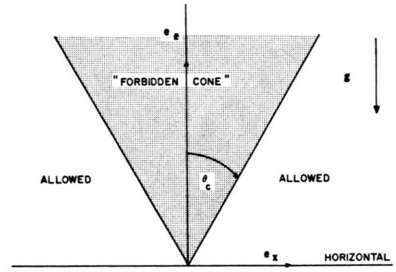

Fig. 12. Forbidden cone for the W_{-} mode.

9. Group and Phase Velocities

Group velocities are easily calculated from the dispersion relation (6.13)

$$V_{\mathrm{G}\pm} = \partial W_{\pm}/\partial\varkappa$$

$$V_{\mathrm{G}\pm} = \frac{\varkappa}{2\,W_{\pm}} \left[1 \pm \frac{\varkappa^{2} + \frac{1}{2}[\frac{1}{2}\gamma^{2} - 4(\gamma - 1)\sin^{2}\theta]}{[\varkappa^{4} + \varkappa^{2}\{\frac{1}{2}\gamma^{2} - 4(\gamma - 1)\sin^{2}\theta\} + \frac{1}{16}\gamma^{4}]^{\frac{1}{2}}}\right]. \qquad (9.1)$$

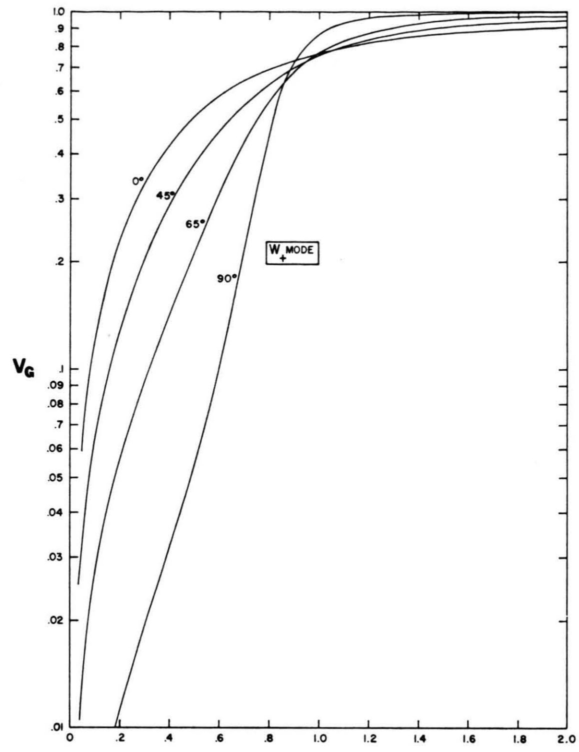

Fig. 13. Group velocities for monatomic gases (W_+ mode).

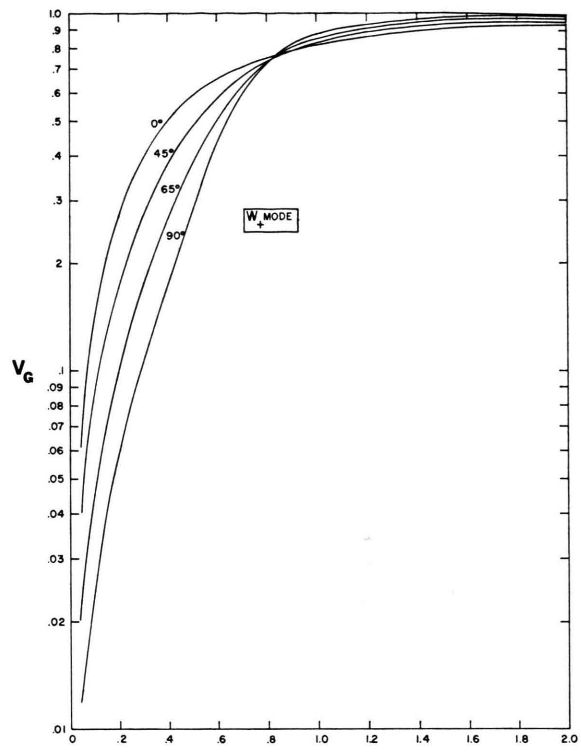

Fig. 15. Group velocities for diatomic gases (W_+ mode).

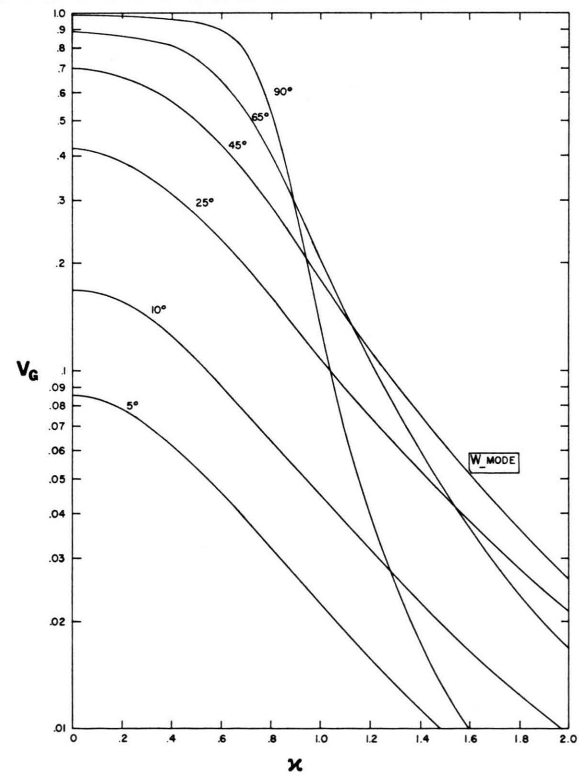

Fig. 14. Group velocities for monatomic gases (W_- mode).

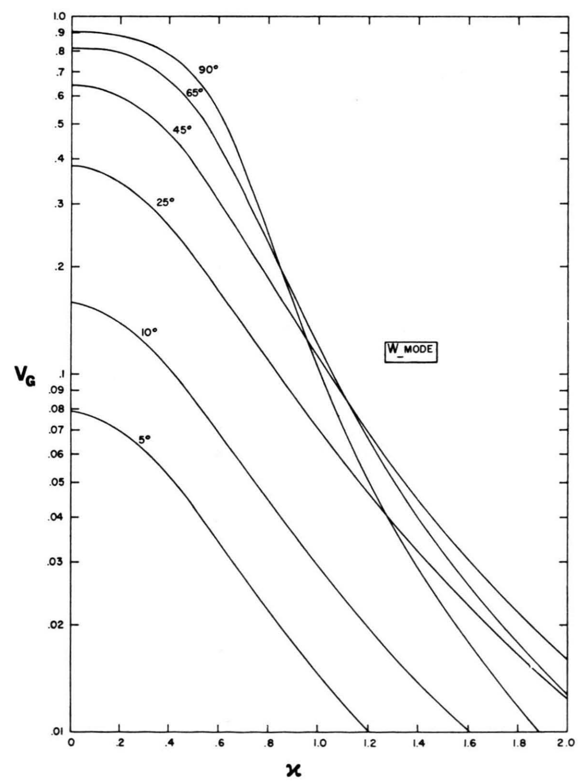

Fig. 16. Group velocities for diatomic gases (W_- mode).

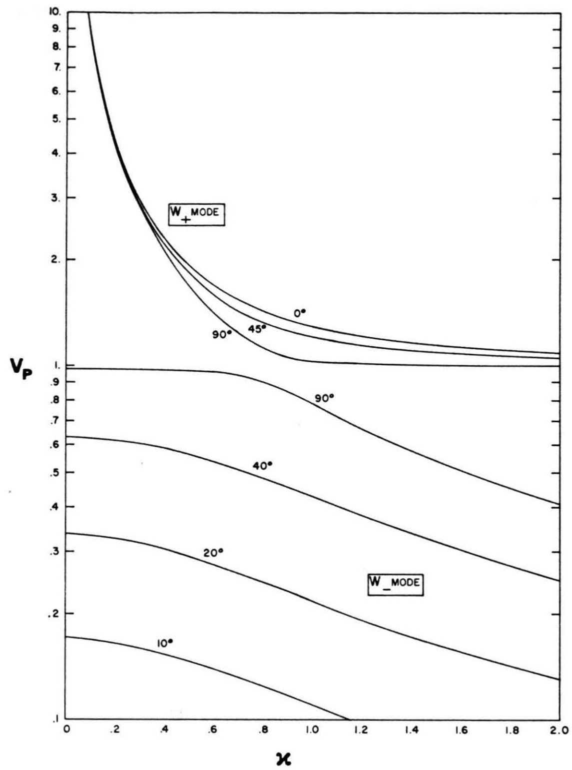

Fig. 17. Phase velocities for monatomic gases.

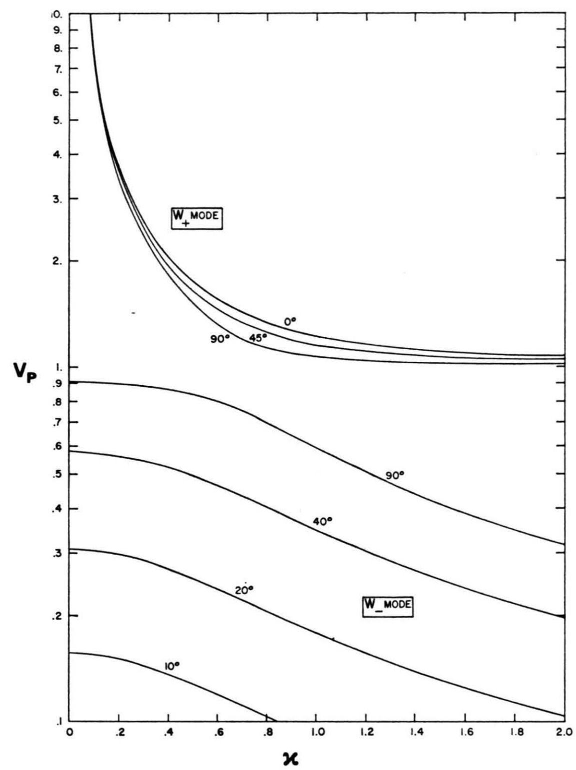

Fig. 19. Phase velocities for diatomic gases.

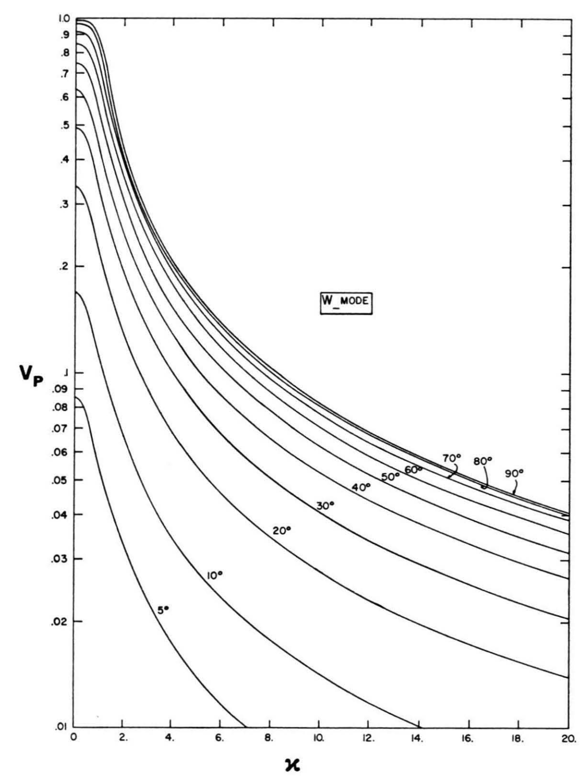

Fig. 18. Phase velocities for monatomic gases (W_- mode).

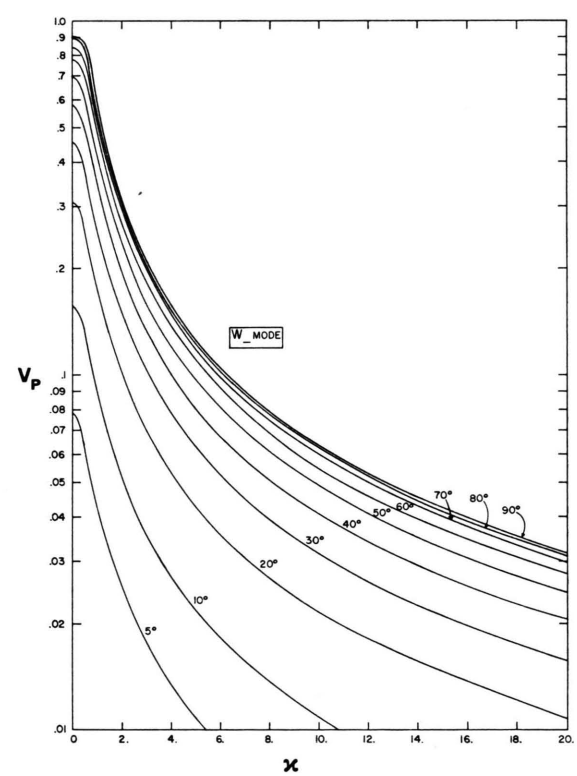

Fig. 20. Phase velocities for diatomic gases (W_- mode).

Graphs of group velocities as a function of the wave number, $\varkappa$, are plotted for various angles in Figs. 13 and 14 for monatomic gases and in Figs. 15 and 16 for diatomic gases.

Phase velocities are calculated directly from Eq. (6.13)

$$V_{P\pm} = W_{\pm}/\varkappa\,, \tag{9.2}$$

and they are graphed as a function of $\varkappa$ for various angles in Figs. 17 and 18 for monatomic gases and in Figs. 19 and 20 for diatomic gases.

V_G and V_P are dimensionless quantities in units of the sound velocity a.

10. Particle Orbits

In this section we consider the orbit of an individual particle in the medium undergoing wave motion. The particle orbits can be obtained from Eq. (5.2) for $\boldsymbol{v}$, respectively for its two components, the two homogeneous Eqs. (6.1) and (6.2). Putting these two equations into a form with the appropriate angle dependence (6.8), we have

$$v_x(\omega^2 - a^2 k^2 \sin^2\theta) + v_z[-a^2 k^2 \sin\theta\cos\theta + i g(\gamma-1) k \sin\theta] = 0\,, \tag{10.1}$$

$$v_x(i g k \sin\theta - a^2 k^2 \sin\theta\cos\theta) + v_z(\omega^2 - a^2 k^2\cos^2\theta + i g \gamma k \cos\theta) = 0\,. \tag{10.2}$$

To obtain the particle orbits, we assume the z-component of the velocity has the form

$$v_z = A \exp\{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\}\,, \tag{10.3}$$

where A is a constant.

From Eq. (10.1) we have the relation

$$v_x = \frac{a^2 k^2 \sin\theta\cos\theta - i g(\gamma-1) k\sin\theta}{\omega^2 - a^2 k^2}\,. \tag{10.4}$$

For small deviations of the particles from their equilibrium positions $\boldsymbol{r}_0 = x_0\,\boldsymbol{e}_x + z_0\,\boldsymbol{e}_z$, we may put

$$x = x_0 + \xi\,, \quad z = z_0 + \zeta \tag{10.5}$$

or respectively,

$$\xi = x - x_0\,, \quad \zeta = z - z_0\,,$$

where ξ and ζ are considered to be small quantities.

By substituting these expressions into Eqs. (10.3) and (10.4), we have after integrating (approximately) for the real parts,

$$z - z_0 = -(A/\omega)\sin(\boldsymbol{k}\cdot\boldsymbol{r}_0 - \omega t) \tag{10.6}$$

and

$$x - x_0 = \left[\frac{(A/\omega)\,k\sin\theta}{\omega^2 - a^2 k^2\sin^2\theta}\right][g(\gamma-1)\cos(\boldsymbol{k}\cdot\boldsymbol{r}_0 - \omega t) - a^2 k\cos\theta\sin(\boldsymbol{k}\cdot\boldsymbol{r}_0 - \omega t)]\,. \tag{10.7}$$

Eqs. (10.6) and (10.7) are the equations for the particle orbits in parametric form, where time is the parameter. Introducing the phase $\varphi_0 \equiv \boldsymbol{k}\cdot\boldsymbol{r}_0 - \omega t$, we have for Eqs. (10.6) and (10.7)

$$\zeta = -(A/\omega)\sin\varphi_0\,, \tag{10.8}$$

$$\xi = \left[\frac{(A/\omega)\,k\sin\theta}{\omega^2 - a^2 k^2\sin^2\theta}\right][g(\gamma-1)\cos\varphi_0 - a^2 k\cos\theta\sin\varphi_0]\,. \tag{10.9}$$

Eliminating φ_0 from Eqs. (10.8) and (10.9), one obtains

$$\xi = \left[\frac{(A/\omega)\,k\sin\theta}{\omega^2 - a^2 k^2\sin^2\theta}\right][g(\gamma-1)\,(1 - (\omega^2/A^2)\,\zeta^2)^{1/2} + a^2 k(\omega/A)\,\zeta\cos\theta]\,. \tag{10.10}$$

This is the orbit equation in nonparametric form which may be rewritten as follows:

$$\frac{\omega^2}{A^2}\frac{(\omega^2 - a^2 k^2\sin^2\theta)^2}{g^2(\gamma-1)^2 k^2\sin^2\theta}\,\xi^2 - 2\,\frac{\omega^2}{A^2}\,\frac{a^2}{g^2(\gamma-1)^2}\,\frac{\omega^2 - a^2 k^2\sin^2\theta}{\tan\theta}\,\xi\zeta + \frac{\omega^2}{A^2}\left[\frac{a^4 k^2\cos^2\theta}{g^2(\gamma-1)^2} + 1\right]\zeta^2 = 1\,. \tag{10.11}$$

We may consider the equilibrium point to be at the origin without modifying the shape of the particle's path. So, let $x_0 = z_0 = 0$, which yields

$$\xi = x\,, \quad \zeta = z\,.$$

Eq. (10.11) then becomes

$$\alpha x^2 - 2\,\eta\, x z + \beta z^2 = 1\,, \tag{10.12}$$

where

$$\alpha = \frac{\omega^2}{A^2}\,\frac{(\omega^2 - a^2 k^2\sin^2\theta)^2}{g^2(\gamma-1)^2 k^2\sin^2\theta}\,,$$

$$\eta = \frac{\omega^2}{A^2}\,\frac{a^2}{g^2(\gamma-1)^2}\,\frac{\omega^2 - a^2 k^2\sin^2\theta}{\tan\theta}\,, \tag{10.13}$$

$$\beta = \frac{\omega^2}{A^2}\left[\frac{a^4 k^2\cos^2\theta}{g^2(\gamma-1)^2} + 1\right].$$

Eq. (10.12) is the equation of an ellipse, but not one aligned along its principal axes; however, it may but into the familiar form of an ellipse by means of a principal axis transformation, which in this case is a rotation, so that the mixed (xz) term in Eq. (10.12) vanishes. Consider the transformation

$$x = x' \cos\psi - z' \sin\psi$$
$$z = x' \sin\psi + z' \cos\psi\,, \qquad (10.14)$$

where (x', z') are the new axes, and ψ is the angle of rotation. By substituting Eqs. (10.14) into Eq. (10.12) and rearranging, we have

$$\alpha' x'^2 - 2\eta' x' z' + \beta' z'^2 = 1\,, \qquad (10.15)$$

where

$$\alpha' \equiv \alpha \cos^2\psi - 2\eta \sin\psi \cos\psi + \beta \sin^2\psi\,,$$
$$\eta' \equiv (\alpha - \beta)\sin\psi\cos\psi + \eta(\cos^2\psi - \sin^2\psi)\,, \qquad (10.16)$$
$$\beta' \equiv \alpha \sin^2\psi + 2\eta\sin\psi\cos\psi + \beta\cos^2\psi\,.$$

Eq. (10.15) is in the familiar form of an ellipse if

$$\eta' = 0;$$

that is, if

$$\tan 2\psi = 2\eta/(\beta - \eta)\,. \qquad (10.17)$$

Hence, Eq. (10.15) becomes

$$\frac{x'^2}{1/\alpha'} + \frac{z'^2}{1/\beta'} = 1\,, \qquad (10.18)$$

the equation of an ellipse.

From Eq. (10.10) one can show that for a wave propagating in the horizontal direction $(\theta = \pi/2)$ the principal axes of the ellipse are oriented along the z and x axis. For a wave propagating in the vertical direction $(\theta = 0)$, one can see directly from Eq. (10.6) and (10.7) that the ellipse degenerates into a straight line along the z axis. For waves propagating at an arbitrary angle θ, the orientation of the ellipse follows from Eq. (10.17). Because the parameters α, β, and η depend on ω and k, the orientation of the ellipse in general depends on the frequency, and it is not aligned along the vector $\boldsymbol{k}$.

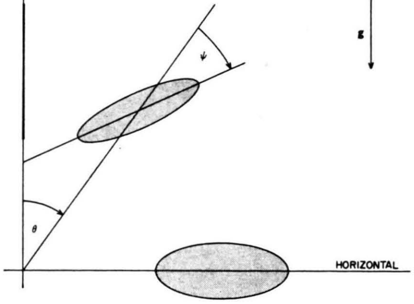

Fig. 21. Particle orbits. For wave propagation in the vertical direction, the elliptic orbits degenerate into a straight line indicating longitudinal waves only.

11. Justification for the Isothermal Assumption

The assumption of an isothermal atmosphere, which has been made in this study, requires a few remarks.

First, in many stratifications of the earth's atmosphere, the assumption of an isothermal atmosphere is rather well satisfied[6], especially in the upper atmosphere above 200 km.

Second, we would like to show that in the more general case of a polytropic atmosphere, where the pressure p as a function of the density ϱ can be expressed by

$$p/p_0 = (\varrho/\varrho_0)^n \qquad (11.1)$$

our results can be applied over large regions with sufficient accuracy. In Eq. (11.1), n is called the polytropic exponent, and it has the value 1 for an isothermal atmosphere.

In a constant gravitational field, the functional dependence of pressure and density on z at equilibrium is given by[7]

$$p_{\mathrm{eq}} = p_0\left(1 - \frac{n-1}{n}\,\frac{\varrho_0}{p_0}\,g\,z\right)^{n/(n-1)} \qquad (11.2)$$

and

$$\varrho_{\mathrm{eq}} = \varrho_0\left(1 - \frac{n-1}{n}\,\frac{\varrho_0}{p_0}\,g\,z\right)^{1/(n-1)}. \qquad (11.3)$$

The isothermal case (2.7) can be obtained from these two equations by making the limiting transition to $n \to 1$ and using the well-known formula

$$e^A = \lim_{m\to\infty}(1 + A/m)^m\,.$$

We consider the perturbing quantities small compared to the equilibrium quantities, as was done for the isothermal case in section 3. Thus, we obtain in first order

$$p = (p_0 + p')\left(1 - \frac{n-1}{n}\,\frac{\varrho_0}{p_0}\,g\,z\right)^{n/(n-1)} \qquad (11.4)$$

and

$$\varrho = (\varrho_0 + \varrho')\left(1 - \frac{n-1}{n}\,\frac{\varrho_0}{p_0}\,g\,z\right)^{1/(n-n)}. \qquad (11.5)$$

[6] H. Wexler and J. E. Caskey (Editors), Rocket and Satellite Meteorology, North-Holland, Amsterdam 1963.

[7] A. Sommerfeld, Mechanics of Deformable Bodies, Academic Press, London 1964.

By substituting Eqs. (11.4) and (11.5) into the Euler equation (2.1), we have

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\left(1 - \frac{n-1}{n}\frac{\varrho_0}{p_0} g z\right)^n \frac{\nabla p'}{\varrho_0} + \frac{p'}{h\,\varrho_0}\boldsymbol{e}_z + \frac{\varrho'}{\varrho_0}\boldsymbol{g},$$

which reduces to Eq. (3.4) for $n = 1$.

The upper boundary of a polytropic atmosphere, where the pressure and density are zero, is given by

$$h_n = \frac{n}{n-1}\frac{p_0}{\varrho_0 g}. \qquad (11.7)$$

This boundary is infinite for the isothermal case ($n = 1$).

If we limit ourselves to regions for which $z \ll h_n$, then we can consider

$$(1 - z/h_n)^n \cong 1$$

for all n. Thus, our results are also valid for a polytropic atmosphere of any n, if they are not applied near the upper boundary of the atmosphere.

12. Conclusion

We have considered the propagation of small amplitude internal acoustic gravity waves in a compressible nonviscous and nonheatconducting stratified atmosphere. The dispersion relation has been derived under the assumption that the fluid behaves as an ideal gas. The dispersion relation was formulated in a non-dimensional form which simplifies the analysis for these waves. Graphs in non-dimensional presentation have been compiled. These graphs may be useful for the study of acoustic gravity waves in planetary and stellar atmospheres.

In the important case of the solar corona, the inclusion of effects arising from magnetic fields should be taken into account. In our present analysis these effects have been omitted and shall be treated in a later paper.

Acknowledgements

Part of this work was carried out at the Institut für Kernenergetik der Universität Stuttgart, Germany, while the authors were on leave from the Desert Research Institute and Department of Physics at the University of Nevada. We wish to thank Professor Dr. K. H. Höcker for the kind hospitality during our stay. Also, part of this work was carried out with the support of the National Aeronautics and Space Administration, U.S.A., while the authors were at the International Centre for Theoretical Physics of the International Atomic Energy Agency in Trieste, Italy. Special thanks are given to Professor Abdus Salam and the I.A.E.A. for the kind hospitality during our stay.